Phase Reduction Method for Strongly Perturbed Limit-Cycle Oscillators - Supplementary Information -

Wataru Kurebayashi,* Sho Shirasaka, and Hiroya Nakao

Graduate School of Information Science and Engineering,

Tokyo Institute of Technology, O-okayama 2-12, Meguro, Tokyo 152-8552, Japan

(Dated: October 22, 2013)

Abstract

This Supplementary Information provides further details of the generalized phase reduction method for strongly perturbed limit-cycle oscillators. A full derivation of the generalized phase equation is given with estimation of the error terms, which takes into account the effect of amplitude relaxation dynamics of the oscillator. Relations between different sensitivity functions are also derived. Accuracy and robustness of the proposed method are examined by numerical simulations for various parameter conditions. Furthermore, analysis of the phase-locking dynamics of a neural oscillator to strong periodic forcing, exhibiting smooth or relaxation oscillations, is also presented.

* kurebayashi.w.aa@m.titech.ac.jp

I. DERIVATION OF THE GENERALIZED PHASE EQUATION

In this section, we give a detailed derivation of the generalized phase equation (2) in the main article, which takes into account the effect of amplitude relaxation of the oscillator state to the cylinder of limit cycles C. Our aim is to evaluate the order of error terms in the generalized phase equation (2). Our argument here is based on a formulation similar to Ref. [1] by Goldobin *et al.*, in which the effect of colored noise on limit-cycle oscillators is analyzed and an effective phase equation that accurately describes the oscillator state is derived by incorporating the effect of amplitude relaxation of the oscillator state to the unperturbed limit-cycle orbit.

As in the main article, we consider a limit-cycle oscillator whose dynamics depends on a time-varying parameter $\boldsymbol{I}(t)$ representing general perturbations, described by

$$\dot{\boldsymbol{X}}(t) = \boldsymbol{F}(\boldsymbol{X}(t), \boldsymbol{I}(t)). \tag{S.1}$$

For simplicity, we assume that the state variable $\boldsymbol{X}(t)$ is two-dimensional ($n = 2$), but the formulation can be straightforwardly extended to higher-dimensional cases.

Suppose that the parameter $\boldsymbol{I}$ is constant for the moment. As explained in the main article, we introduce an extended phase space $\mathbb{R}^n \times A$ and define a generalized phase θ and amplitude r as functions of $(\boldsymbol{X}, \boldsymbol{I})$ in U. Here, r gives the distance of the oscillator state $\boldsymbol{X}$ from the unperturbed stable limit cycle $\boldsymbol{X}_0(\theta, \boldsymbol{I})$. For each constant value of $\boldsymbol{I} \in A$, as argued in the Supplementary Information of Ref. [1], we can define a phase $\theta = \Theta(\boldsymbol{X}, \boldsymbol{I})$ and an amplitude $r = R(\boldsymbol{X}, \boldsymbol{I})$ such that

$$\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}} \cdot \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{I}) = \omega(\boldsymbol{I}), \tag{S.2}$$

$$\frac{\partial R(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}} \cdot \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{I}) = -\lambda(\boldsymbol{I}) R(\boldsymbol{X}, \boldsymbol{I}), \tag{S.3}$$

where $\lambda(\boldsymbol{I})$ is the absolute value of the second Floquet exponent of Eq. (S.1) for each $\boldsymbol{I}$. We further assume that $\Theta(\boldsymbol{X}, \boldsymbol{I})$ and $R(\boldsymbol{X}, \boldsymbol{I})$ are continuously differentiable with respect to $\boldsymbol{X}$ and $\boldsymbol{I}$. Equations (S.2) and (S.3) guarantee that

$$\dot{\theta} = \omega(\boldsymbol{I}), \quad \dot{r} = -\lambda(\boldsymbol{I}) r \tag{S.4}$$

always hold for each $\boldsymbol{I}$. In the absence of perturbations, the amplitude $r = R(\boldsymbol{X}, \boldsymbol{I})$ decays to 0 exponentially, and the phase $\theta = \Theta(\boldsymbol{X}, \boldsymbol{I})$ increases constantly.

Now we suppose that the parameter $\boldsymbol{I}(t)$ can vary with time. As explained in the main article, we decompose the parameter $\boldsymbol{I}(t)$ into a slowly varying component $\boldsymbol{q}(\epsilon t)$ and remaining weak

fluctuations $\sigma \boldsymbol{p}(t)$ as $\boldsymbol{I}(t) = \boldsymbol{q}(\epsilon t) + \sigma \boldsymbol{p}(t)$. We define a phase $\theta(t)$ and an amplitude $r(t)$ of the oscillator as follows:

$$\theta(t) = \Theta(\boldsymbol{X}(t), \boldsymbol{q}(\epsilon t)), \tag{S.5}$$

$$r(t) = R(\boldsymbol{X}(t), \boldsymbol{q}(\epsilon t)). \tag{S.6}$$

Since $\Theta(\boldsymbol{X}, \boldsymbol{I})$ and $R(\boldsymbol{X}, \boldsymbol{I})$ are continuously differentiable with respect to $\boldsymbol{X}$ and $\boldsymbol{I}$, we can derive the dynamical equations for $\theta(t)$ and $r(t)$ as

$$\dot{\theta} = \left.\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}}\right|_{(\boldsymbol{X}, \boldsymbol{q}(\epsilon t))} \cdot \frac{d\boldsymbol{X}(t)}{dt} + \left.\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{I}}\right|_{(\boldsymbol{X}, \boldsymbol{q}(\epsilon t))} \cdot \frac{d\boldsymbol{q}(\epsilon t)}{dt}, \tag{S.7}$$

$$\dot{r} = \left.\frac{\partial R(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}}\right|_{(\boldsymbol{X}, \boldsymbol{q}(\epsilon t))} \cdot \frac{d\boldsymbol{X}(t)}{dt} + \left.\frac{\partial R(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{I}}\right|_{(\boldsymbol{X}, \boldsymbol{q}(\epsilon t))} \cdot \frac{d\boldsymbol{q}(\epsilon t)}{dt}. \tag{S.8}$$

Plugging $\boldsymbol{I}(t) = \boldsymbol{q}(\epsilon t) + \sigma \boldsymbol{p}(t)$ into Eq. (S.1) and expanding it to the first order in σ, we can derive

$$\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{q}(\epsilon t)) + \sigma \boldsymbol{G}(\boldsymbol{X}, \boldsymbol{q}(\epsilon t)) \boldsymbol{p}(t) + O(\sigma^2), \tag{S.9}$$

where the matrix $\boldsymbol{G}$ is defined in the main article. Substituting Eqs. (S.2), (S.3), and (S.9) into Eqs. (S.7) and (S.8), we can obtain

$$\dot{\theta} = \omega(\boldsymbol{q}(\epsilon t)) + \sigma \left.\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}}\right|_{(\boldsymbol{X}, \boldsymbol{q}(\epsilon t))} \cdot \boldsymbol{G}(\boldsymbol{X}, \boldsymbol{q}(\epsilon t)) \boldsymbol{p}(t) + \epsilon \left.\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{I}}\right|_{(\boldsymbol{X}, \boldsymbol{q}(\epsilon t))} \cdot \dot{\boldsymbol{q}}(\epsilon t) + O(\sigma^2), \tag{S.10}$$

$$\dot{r} = -\lambda(\boldsymbol{q}(\epsilon t)) r + \sigma \left.\frac{\partial R(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}}\right|_{(\boldsymbol{X}, \boldsymbol{q}(\epsilon t))} \cdot \boldsymbol{G}(\boldsymbol{X}, \boldsymbol{q}(\epsilon t)) \boldsymbol{p}(t) + \epsilon \left.\frac{\partial R(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{I}}\right|_{(\boldsymbol{X}, \boldsymbol{q}(\epsilon t))} \cdot \dot{\boldsymbol{q}}(\epsilon t) + O(\sigma^2), \tag{S.11}$$

where $\dot{\boldsymbol{q}}(\epsilon t)$ denotes $d\boldsymbol{q}(\epsilon t)/d(\epsilon t)$. For simplicity of notation, we define $\boldsymbol{\zeta}_\theta(\theta, r, \boldsymbol{I}) \in \mathbb{R}^m$, $\boldsymbol{\zeta}_r(\theta, r, \boldsymbol{I}) \in \mathbb{R}^m$, $\boldsymbol{\xi}_\theta(\theta, r, \boldsymbol{I}) \in \mathbb{R}^m$ and $\boldsymbol{\xi}_r(\theta, r, \boldsymbol{I}) \in \mathbb{R}^m$, respectively, as

$$\boldsymbol{\zeta}_\theta(\theta, r, \boldsymbol{I}) = \boldsymbol{G}(\boldsymbol{X}, \boldsymbol{I})^\top \left.\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}}\right|_{\boldsymbol{X} = \boldsymbol{X}(\theta, r, \boldsymbol{I})}, \tag{S.12}$$

$$\boldsymbol{\zeta}_r(\theta, r, \boldsymbol{I}) = \boldsymbol{G}(\boldsymbol{X}, \boldsymbol{I})^\top \left.\frac{\partial R(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}}\right|_{\boldsymbol{X} = \boldsymbol{X}(\theta, r, \boldsymbol{I})}, \tag{S.13}$$

$$\boldsymbol{\xi}_\theta(\theta, r, \boldsymbol{I}) = \left.\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{I}}\right|_{\boldsymbol{X} = \boldsymbol{X}(\theta, r, \boldsymbol{I})}, \tag{S.14}$$

$$\boldsymbol{\xi}_r(\theta, r, \boldsymbol{I}) = \left.\frac{\partial R(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{I}}\right|_{\boldsymbol{X} = \boldsymbol{X}(\theta, r, \boldsymbol{I})}, \tag{S.15}$$

where $\boldsymbol{X}(\theta, r, \boldsymbol{I}) \in \mathbb{R}^2$ represents an oscillator state with $\theta = \Theta(\boldsymbol{X}, \boldsymbol{I})$, $r = R(\boldsymbol{X}, \boldsymbol{I})$, and parameter $\boldsymbol{I}$. Using Eqs. (S.12), (S.13), (S.14), and (S.15), we can rewrite Eqs. (S.10) and (S.11) as

$$\dot{\theta} = \omega(\boldsymbol{q}(\epsilon t)) + \sigma \boldsymbol{\zeta}_\theta(\theta, r, \boldsymbol{I}) \cdot \boldsymbol{p}(t) + \epsilon \boldsymbol{\xi}_\theta(\theta, r, \boldsymbol{I}) \cdot \dot{\boldsymbol{q}}(\epsilon t) + O(\sigma^2), \tag{S.16}$$

$$\dot{r} = -\lambda(\boldsymbol{q}(\epsilon t)) r + \sigma \boldsymbol{\zeta}_r(\theta, r, \boldsymbol{I}) \cdot \boldsymbol{p}(t) + \epsilon \boldsymbol{\xi}_r(\theta, r, \boldsymbol{I}) \cdot \dot{\boldsymbol{q}}(\epsilon t) + O(\sigma^2). \tag{S.17}$$

Note that $\boldsymbol{\zeta}_\theta(\theta, 0, \boldsymbol{I})$ and $\boldsymbol{\xi}_\theta(\theta, 0, \boldsymbol{I})$ are equivalent to the sensitivity functions $\boldsymbol{\zeta}(\theta, \boldsymbol{I})$ and $\boldsymbol{\xi}(\theta, \boldsymbol{I})$ defined in the main article. The functions $\boldsymbol{\zeta}_r(\theta, r, \boldsymbol{I})$ and $\boldsymbol{\xi}_r(\theta, r, \boldsymbol{I})$ represent sensitivities of the amplitude to the small fluctuations and to the slowly varying component of the applied perturbations, respectively. In the main article, we also assumed that $\boldsymbol{q}(\epsilon t)$ varies sufficiently slowly as compared to the relaxation time of perturbed orbits to C. By using the absolute value of the Floquet exponent $\lambda(\boldsymbol{I})$ and the slowly varying component $\boldsymbol{q}(\epsilon t)$, this assumption can be written as

$$\epsilon \ll \lambda(\boldsymbol{q}(\epsilon t)), \quad \text{or} \quad \frac{\epsilon}{\lambda(\boldsymbol{q}(\epsilon t))} \ll 1. \tag{S.18}$$

Now, we show that the following relation between the sensitivity functions for the amplitude holds:

$$\boldsymbol{\xi}_r(\theta, 0, \boldsymbol{I}) = -\frac{1}{\omega(\boldsymbol{I})} \int_0^\infty e^{-\lambda(\boldsymbol{I})\phi/\omega(\boldsymbol{I})} \boldsymbol{\zeta}_r(\theta - \phi, 0, \boldsymbol{I}) d\phi \tag{S.19}$$

$$= -\frac{1}{\lambda(\boldsymbol{I})} \int_0^\infty e^{-s} \boldsymbol{\zeta}_r(\theta - \omega(\boldsymbol{I})s/\lambda(\boldsymbol{I}), 0, \boldsymbol{I}) ds, \tag{S.20}$$

where we defined $s = \lambda(\boldsymbol{I})\phi/\omega(\boldsymbol{I})$ in the second line. From Eq. (S.3),

$$\frac{\partial R(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}} \cdot \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{I}) = -\lambda(\boldsymbol{I}) R(\boldsymbol{X}, \boldsymbol{I}) \tag{S.21}$$

holds. We differentiate Eq. (S.21) with respect to $\boldsymbol{I}$ and plug in $\boldsymbol{X} = \boldsymbol{X}_0(\theta, \boldsymbol{I})$. Then, from the left-hand side of Eq. (S.21), we obtain

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{I}} \left[\frac{\partial R(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}} \cdot \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{I}) \right] \Bigg|_{\boldsymbol{X} = \boldsymbol{X}_0(\theta, \boldsymbol{I})} &= \left[\frac{\partial}{\partial \boldsymbol{I}} \left(\frac{\partial R(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}} \right) \right]^\top \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{I}) \Bigg|_{\boldsymbol{X} = \boldsymbol{X}_0(\theta, \boldsymbol{I})} \\
&\quad + \frac{\partial \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{I}}^\top \frac{\partial R(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}} \Bigg|_{\boldsymbol{X} = \boldsymbol{X}_0(\theta, \boldsymbol{I})} \\
&= \left[\frac{\partial}{\partial \boldsymbol{X}} \left(\frac{\partial R(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{I}} \right) \right] \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{I}) \Bigg|_{\boldsymbol{X} = \boldsymbol{X}_0(\theta, \boldsymbol{I})} + \boldsymbol{\zeta}_r(\theta, 0, \boldsymbol{I}),
\end{aligned} \tag{S.22}$$

where $\partial/\partial \boldsymbol{I}$ denotes a differential operator defined as $(\partial/\partial \boldsymbol{I}) f(\boldsymbol{I}) = [\partial f(\boldsymbol{I})/\partial I_1, \ldots, \partial f(\boldsymbol{I})/\partial I_m]^\top \in \mathbb{R}^m$ for a scalar function $f(\boldsymbol{I})$, $\frac{\partial}{\partial \boldsymbol{I}}(\frac{\partial R(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}})$ is a matrix whose (i, j)-th element is given by $\frac{\partial^2 R(\boldsymbol{X}, \boldsymbol{I})}{\partial X_i \partial I_j}$, and $\frac{\partial}{\partial \boldsymbol{X}}(\frac{\partial R(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{I}})$ is the transpose of $\frac{\partial}{\partial \boldsymbol{I}}(\frac{\partial R(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}})$. Here, the first term of the right-hand side of

Eq. (S.22) can be written as

$$\left[\frac{\partial}{\partial \boldsymbol{X}}\left(\frac{\partial R(\boldsymbol{X},\boldsymbol{I})}{\partial \boldsymbol{I}}\right)\right]\boldsymbol{F}(\boldsymbol{X},\boldsymbol{I})\bigg|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta,\boldsymbol{I})} = \left[\frac{\partial}{\partial \boldsymbol{X}}\left(\frac{\partial R(\boldsymbol{X},\boldsymbol{I})}{\partial \boldsymbol{I}}\right)\right]\bigg|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta,\boldsymbol{I})}\frac{d\boldsymbol{X}_0(\omega(\boldsymbol{I})t,\boldsymbol{I})}{dt}\bigg|_{t=\theta/\omega(\boldsymbol{I})}$$
$$=\omega(\boldsymbol{I})\left[\frac{\partial}{\partial \boldsymbol{X}}\left(\frac{\partial R(\boldsymbol{X},\boldsymbol{I})}{\partial \boldsymbol{I}}\right)\right]\bigg|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta,\boldsymbol{I})}\frac{\partial \boldsymbol{X}_0(\theta,\boldsymbol{I})}{\partial\theta} = \omega(\boldsymbol{I})\frac{\partial}{\partial\theta}\left(\frac{\partial R(\boldsymbol{X},\boldsymbol{I})}{\partial \boldsymbol{I}}\right)\bigg|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta,\boldsymbol{I})}$$
$$=\omega(\boldsymbol{I})\frac{\partial \boldsymbol{\xi}_r(\theta,0,\boldsymbol{I})}{\partial\theta}. \tag{S.23}$$

Furthermore, differentiating the right-hand side of Eq. (S.21), we can derive

$$\frac{\partial}{\partial \boldsymbol{I}}\left[-\lambda(\boldsymbol{I})R(\boldsymbol{X},\boldsymbol{I})\right]\bigg|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta,\boldsymbol{I})} = -\left[\frac{d\lambda(\boldsymbol{I})}{d\boldsymbol{I}}R(\boldsymbol{X},\boldsymbol{I})+\lambda(\boldsymbol{I})\frac{\partial R(\boldsymbol{X},\boldsymbol{I})}{\partial \boldsymbol{I}}\right]\bigg|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta,\boldsymbol{I})}$$
$$= -\lambda(\boldsymbol{I})\boldsymbol{\xi}_r(\theta,0,\boldsymbol{I}), \tag{S.24}$$

where we used $R(\boldsymbol{X}_0(\theta,\boldsymbol{I}),\boldsymbol{I})=0$. Thus, from Eqs. (S.21)–(S.24), we can obtain

$$\omega(\boldsymbol{I})\frac{\partial \boldsymbol{\xi}_r(\theta,0,\boldsymbol{I})}{\partial\theta}+\boldsymbol{\zeta}_r(\theta,0,\boldsymbol{I}) = -\lambda(\boldsymbol{I})\boldsymbol{\xi}_r(\theta,0,\boldsymbol{I}) \tag{S.25}$$

Since Eq. (S.25) is a linear first-order ordinary differential equation for $\boldsymbol{\xi}_r(\theta,0,\boldsymbol{I})$, this equation can be solved as follows:

$$\boldsymbol{\xi}_r(\theta,0,\boldsymbol{I}) = -\frac{1}{\omega(\boldsymbol{I})}\int_{-\infty}^{\theta} e^{\lambda(\boldsymbol{I})(\theta'-\theta)/\omega(\boldsymbol{I})}\boldsymbol{\zeta}_r(\theta',0,\boldsymbol{I})d\theta', \tag{S.26}$$

which leads to Eqs. (S.19) and (S.20).

Using the derived Eq. (S.20), we can estimate the order of $\boldsymbol{\xi}_r(\theta,0,\boldsymbol{q}(\epsilon t))$ as

$$\boldsymbol{\xi}_r(\theta,0,\boldsymbol{q}(\epsilon t)) = \frac{1}{\lambda(\boldsymbol{I})}\int_0^{\infty} e^{-s}\boldsymbol{\zeta}_r(\theta-\omega(\boldsymbol{I})s/\lambda(\boldsymbol{I}),0,\boldsymbol{I})ds\bigg|_{\boldsymbol{I}=\boldsymbol{q}(\epsilon t)}$$
$$= \frac{1}{\lambda(\boldsymbol{I})}\int_0^{\infty} e^{-s}\boldsymbol{\zeta}_r(\theta,0,\boldsymbol{I})ds\bigg|_{\boldsymbol{I}=\boldsymbol{q}(\epsilon t)} + O\left(\frac{1}{\lambda(\boldsymbol{q}(\epsilon t))^2}\right)$$
$$= O\left(\frac{1}{\lambda(\boldsymbol{q}(\epsilon t))}\right), \tag{S.27}$$

where we expanded $\boldsymbol{\zeta}_r(\theta,r,\boldsymbol{I})$ in θ in the second line. For simplicity of notation, we introduce $\tilde{\boldsymbol{\xi}}_r(\theta,\boldsymbol{I})$ as follows:

$$\tilde{\boldsymbol{\xi}}_r(\theta,\boldsymbol{I}) = \lambda(\boldsymbol{I})\boldsymbol{\xi}_r(\theta,0,\boldsymbol{I}) = \int_0^{\infty} e^{-s}\boldsymbol{\zeta}_r(\theta-\omega(\boldsymbol{I})s/\lambda(\boldsymbol{I}),0,\boldsymbol{I})ds. \tag{S.28}$$

Note that $\tilde{\boldsymbol{\xi}}_r(\theta,\boldsymbol{I})$ is of the order $O(1)$.

To evaluate the order of $r(t)$, we approximate the solution to Eq. (S.17) describing the oscillator amplitude in a small neighborhood of $t=t'$. We introduce a small parameter $\tilde{\epsilon} := \epsilon/\lambda(\boldsymbol{q}(\epsilon t'))$,

which is sufficiently small ($\ll 1$) by the assumption that $\epsilon \ll \lambda(\boldsymbol{q}(\epsilon t))$. Then, using the small parameters σ and $\tilde{\epsilon}$, we expand the solutions to Eqs. (S.16) and (S.17) as follows:

$$\theta(t) = \theta_0(t) + \sigma\theta_{\sigma,1}(t) + \tilde{\epsilon}\theta_{\epsilon,1}(t) + \cdots, \tag{S.29}$$

$$r(t) = r_0(t) + \sigma r_{\sigma,1}(t) + \tilde{\epsilon} r_{\epsilon,1}(t) + \cdots, \tag{S.30}$$

where $\theta_0(t)$ and $r_0(t)$ are the lowest order solutions and $\theta_{\sigma,j}(t)$, $r_{\sigma,j}(t)$, $\theta_{\epsilon,j}(t)$, and $r_{\epsilon,j}(t)$ are jth order perturbations. The lowest order solutions are given by $\theta_0(t) = \theta(t') + \omega(\boldsymbol{q}(\epsilon t'))(t - t')$ and $r_0(t) = 0$ in the neighborhood of $t = t'$. By introducing a rescaled time $s = \Phi(t) := \int_0^t \lambda(\boldsymbol{q}(\epsilon t'))dt'$ (i.e., $ds = \lambda(\boldsymbol{q}(\epsilon t))dt$), we can rewrite Eq. (S.17) as

$$\frac{dr}{ds} = -r + \frac{\sigma\boldsymbol{\zeta}_r(\theta, r, \boldsymbol{q}(\epsilon t))}{\lambda(\boldsymbol{q}(\epsilon t))} \cdot \boldsymbol{p}(t) + \frac{\epsilon\boldsymbol{\xi}_r(\theta, r, \boldsymbol{q}(\epsilon t))}{\lambda(\boldsymbol{q}(\epsilon t))} \cdot \dot{\boldsymbol{q}}(\epsilon t). \tag{S.31}$$

We also expand $\boldsymbol{q}(\epsilon t)$ around $t = t'$ ($s = \Phi(t')$) as $\boldsymbol{q}(\epsilon t) = \boldsymbol{q}(\epsilon t') + \epsilon\boldsymbol{q}'(\epsilon t')(t - t') + \cdots$. Plugging $\theta(t) = \theta_0(t) + O(\sigma, \tilde{\epsilon})$, $r(t) = r_0(t) + O(\sigma, \tilde{\epsilon})$ and $\boldsymbol{q}(\epsilon t) = \boldsymbol{q}(\epsilon t') + O(\epsilon)$ into Eq. (S.31), we can derive

$$\begin{aligned}
\frac{dr}{ds} &= -r + \sigma\frac{\boldsymbol{\zeta}_r(\theta_0(t) + O(\sigma,\tilde{\epsilon}), 0 + O(\sigma,\tilde{\epsilon}), \boldsymbol{q}(\epsilon t') + O(\epsilon))}{\lambda(\boldsymbol{q}(\epsilon t') + O(\epsilon))} \cdot \boldsymbol{p}(t) \\
&\quad + \epsilon\frac{\boldsymbol{\xi}_r(\theta_0(t) + O(\sigma,\tilde{\epsilon}), 0 + O(\sigma,\tilde{\epsilon}), \boldsymbol{q}(\epsilon t') + O(\epsilon))}{\lambda(\boldsymbol{q}(\epsilon t') + O(\epsilon))} \cdot \dot{\boldsymbol{q}}(\epsilon t) \\
&= -r + \sigma(1 + O(\epsilon))\frac{\boldsymbol{\zeta}_r(\theta_0(t) + O(\sigma,\tilde{\epsilon}), 0 + O(\sigma,\tilde{\epsilon}), \boldsymbol{q}(\epsilon t') + O(\epsilon))}{\lambda(\boldsymbol{q}(\epsilon t'))} \cdot \boldsymbol{p}(t) \\
&\quad + \epsilon(1 + O(\epsilon))\frac{\boldsymbol{\xi}_r(\theta_0(t) + O(\sigma,\tilde{\epsilon}), 0 + O(\sigma,\tilde{\epsilon}), \boldsymbol{q}(\epsilon t') + O(\epsilon))}{\lambda(\boldsymbol{q}(\epsilon t'))} \cdot \dot{\boldsymbol{q}}(\epsilon t) \\
&= -r + \sigma(1 + O(\epsilon))\frac{\boldsymbol{\zeta}_r(\theta_0(t), 0, \boldsymbol{q}(\epsilon t') + O(\epsilon))}{\lambda(\boldsymbol{q}(\epsilon t'))} \cdot \boldsymbol{p}(t) \\
&\quad + \epsilon(1 + O(\epsilon))\frac{\boldsymbol{\xi}_r(\theta_0(t), 0, \boldsymbol{q}(\epsilon t') + O(\epsilon))}{\lambda(\boldsymbol{q}(\epsilon t'))} \cdot \dot{\boldsymbol{q}}(\epsilon t) + O(\sigma^2, \sigma\tilde{\epsilon}, \tilde{\epsilon}^2).
\end{aligned} \tag{S.32}$$

Substituting Eq. (S.28) into the above equation, we obtain

$$\begin{aligned}
\frac{dr}{ds} &= -r + \sigma(1 + O(\epsilon))\frac{\boldsymbol{\zeta}_r(\theta_0(t), 0, \boldsymbol{q}(\epsilon t') + O(\epsilon))}{\lambda(\boldsymbol{q}(\epsilon t'))} \cdot \boldsymbol{p}(t) \\
&\quad + \epsilon(1 + O(\epsilon))\frac{\tilde{\boldsymbol{\xi}}_r(\theta_0(t), \boldsymbol{q}(\epsilon t') + O(\epsilon))}{\lambda(\boldsymbol{q}(\epsilon t'))\lambda(\boldsymbol{q}(\epsilon t') + O(\epsilon))} \cdot \dot{\boldsymbol{q}}(\epsilon t) + O(\sigma^2, \sigma\tilde{\epsilon}, \tilde{\epsilon}^2) \\
&= -r + \sigma(1 + O(\epsilon))\frac{\boldsymbol{\zeta}_r(\theta_0(t), 0, \boldsymbol{q}(\epsilon t')) + O(\epsilon)}{\lambda(\boldsymbol{q}(\epsilon t'))} \cdot \boldsymbol{p}(t) \\
&\quad + \epsilon(1 + O(\epsilon))\frac{\tilde{\boldsymbol{\xi}}_r(\theta_0(t), \boldsymbol{q}(\epsilon t')) + O(\epsilon)}{\lambda(\boldsymbol{q}(\epsilon t'))^2} \cdot \dot{\boldsymbol{q}}(\epsilon t) + O(\sigma^2, \sigma\tilde{\epsilon}, \tilde{\epsilon}^2) \\
&= -r + \sigma\frac{\boldsymbol{\zeta}_r\big(\theta(t') + \omega(\boldsymbol{q}(\epsilon t'))(t - t'), 0, \boldsymbol{q}(\epsilon t')\big)}{\lambda(\boldsymbol{q}(\epsilon t'))} \cdot \boldsymbol{p}(t) \\
&\quad + \epsilon\frac{\tilde{\boldsymbol{\xi}}_r\big(\theta(t') + \omega(\boldsymbol{q}(\epsilon t'))(t - t'), \boldsymbol{q}(\epsilon t')\big)}{\lambda(\boldsymbol{q}(\epsilon t'))^2} \cdot \dot{\boldsymbol{q}}(\epsilon t) + O(\sigma^2, \sigma\tilde{\epsilon}, \tilde{\epsilon}^2).
\end{aligned} \tag{S.33}$$

By integrating Eq. (S.33), we can estimate the order of $r(t')$ as

$$\begin{aligned} r(t') &= \frac{\sigma}{\lambda(\boldsymbol{q}(\epsilon t'))}\int_{-\infty}^{\Phi(t')} e^{s-\Phi(t')}\boldsymbol{\zeta}_r\big(\theta(t')+\omega(\boldsymbol{q}(\epsilon t'))(t-t'),0,\boldsymbol{q}(\epsilon t')\big)\cdot\boldsymbol{p}(t)\big|_{t=\Phi^{-1}(s)}ds \\ &+ \frac{\epsilon}{\lambda(\boldsymbol{q}(\epsilon t'))^2}\int_{-\infty}^{\Phi(t')} e^{s-\Phi(t')}\tilde{\boldsymbol{\xi}}_r\big(\theta(t')+\omega(\boldsymbol{q}(\epsilon t'))(t-t'),\boldsymbol{q}(\epsilon t')\big)\cdot\dot{\boldsymbol{q}}(\epsilon t)\big|_{t=\Phi^{-1}(s)}ds + O(\sigma^2,\sigma\tilde{\epsilon},\tilde{\epsilon}^2) \\ &= O\left(\frac{\sigma}{\lambda(\boldsymbol{q}(\epsilon t'))},\frac{\epsilon}{\lambda(\boldsymbol{q}(\epsilon t'))^2}\right). \end{aligned} \tag{S.34}$$

Now, by expanding Eq. (S.10) in r, we can obtain

$$\begin{aligned} \dot{\theta} &= \omega(\boldsymbol{q}(\epsilon t)) + \sigma\boldsymbol{\zeta}_\theta(\theta,0,\boldsymbol{q}(\epsilon t))\cdot\boldsymbol{p}(t) + \epsilon\boldsymbol{\xi}_\theta(\theta,0,\boldsymbol{q}(\epsilon t))\cdot\dot{\boldsymbol{q}}(\epsilon t) \\ &+\sigma r\frac{\partial\boldsymbol{\zeta}_\theta(\theta,0,\boldsymbol{q}(\epsilon t))}{\partial r}\cdot\boldsymbol{p}(t) + \epsilon r\frac{\partial\boldsymbol{\xi}_\theta(\theta,0,\boldsymbol{q}(\epsilon t))}{\partial r}\cdot\dot{\boldsymbol{q}}(\epsilon t) + O(r^2). \end{aligned} \tag{S.35}$$

Substituting Eq. (S.34) into Eq. (S.35) and neglecting higher order terms in r, we can derive the generalized phase equation (2) in the main article,

$$\begin{aligned} \dot{\theta} &= \omega(\boldsymbol{q}(\epsilon t)) + \sigma\boldsymbol{\zeta}_\theta(\theta,0,\boldsymbol{q}(\epsilon t))\cdot\boldsymbol{p}(t) + O\left(\frac{\sigma^2}{\lambda(\boldsymbol{q}(\epsilon t))},\frac{\sigma\epsilon}{\lambda(\boldsymbol{q}(\epsilon t))^2}\right) \\ &+\epsilon\boldsymbol{\xi}_\theta(\theta,0,\boldsymbol{q}(\epsilon t))\cdot\dot{\boldsymbol{q}}(\epsilon t) + O\left(\frac{\sigma\epsilon}{\lambda(\boldsymbol{q}(\epsilon t))},\frac{\epsilon^2}{\lambda(\boldsymbol{q}(\epsilon t))^2}\right). \end{aligned} \tag{S.36}$$

Equation (S.36) reveals that our phase equation well approximates the exact phase dynamics under the conditions that

$$\frac{\sigma^2}{\lambda(\boldsymbol{q}(\epsilon t))} \ll \sigma,\ \ \frac{\sigma\epsilon}{\lambda(\boldsymbol{q}(\epsilon t))^2} \ll \sigma,\ \ \frac{\sigma\epsilon}{\lambda(\boldsymbol{q}(\epsilon t))} \ll \epsilon,\ \text{ and }\ \frac{\epsilon^2}{\lambda(\boldsymbol{q}(\epsilon t))^2} \ll \epsilon. \tag{S.37}$$

Here, we compared the first two error terms $\frac{\sigma^2}{\lambda(\boldsymbol{q}(\epsilon t))}$ and $\frac{\sigma\epsilon}{\lambda(\boldsymbol{q}(\epsilon t))^2}$ with σ, and the last two $\frac{\sigma\epsilon}{\lambda(\boldsymbol{q}(\epsilon t))}$ and $\frac{\epsilon^2}{\lambda(\boldsymbol{q}(\epsilon t))^2}$ with ϵ, because the first and last two error terms arose when we expanded the second term $\sigma\boldsymbol{\zeta}_\theta(\theta,r,\boldsymbol{q}(\epsilon t))\cdot\boldsymbol{p}(t)$ $(=O(\sigma))$ and the third term $\epsilon\boldsymbol{\xi}_\theta(\theta,r,\boldsymbol{q}(\epsilon t))\cdot\dot{\boldsymbol{q}}(\epsilon t)$ $(=O(\epsilon))$ of Eq. (S.16) in r, respectively. These conditions are satisfied when

$$\frac{\epsilon}{\lambda(\boldsymbol{q}(\epsilon t))^2} \ll 1 \ \text{ and }\ \frac{\sigma}{\lambda(\boldsymbol{q}(\epsilon t))} \ll 1, \tag{S.38}$$

namely, when (i) the timescale of the slowly varying component $\boldsymbol{q}(\epsilon t)$ is much larger than the relaxation time of perturbed orbits to C, and (ii) the remaining fluctuations $\sigma\boldsymbol{p}(t)$ is sufficiently weak, as we assumed in the main article.

For limit-cycle oscillators with higher-dimensional state variables ($n \geq 3$), we can also derive a phase equation corresponding to Eq. (S.36). In higher-dimensional cases, the system of Eq. (S.1) has n (≥ 3) Floquet exponents. Let $\lambda_j(\boldsymbol{I})$ denote the absolute value of the j-th largest Floquet exponent of the oscillator for a given constant $\boldsymbol{I}$ ($\lambda_1(\boldsymbol{I}) = 0 > \lambda_2(\boldsymbol{I}) \geq \cdots \geq \lambda_n(\boldsymbol{I})$). In these

exponents, the second largest exponent $\lambda_2(\boldsymbol{I})$ dominates the relaxation time of perturbed orbits. Thus, using the absolute value of the second largest Floquet exponent $\lambda_2(\boldsymbol{q}(\epsilon t))$ instead of $\lambda(\boldsymbol{q}(\epsilon t))$, we can obtain the same results as Eq. (S.36); that is, we can obtain the following phase equation also for the higher-dimensional cases ($n \geq 3$):

$$\begin{aligned}\dot{\theta} &= \omega(\boldsymbol{q}(\epsilon t)) + \sigma \boldsymbol{\zeta}_\theta(\theta, 0, \boldsymbol{q}(\epsilon t)) \cdot \boldsymbol{p}(t) + \epsilon \boldsymbol{\xi}_\theta(\theta, 0, \boldsymbol{q}(\epsilon t)) \cdot \dot{\boldsymbol{q}}(\epsilon t) \\ &\quad + O\left(\frac{\epsilon^2}{\lambda_2(\boldsymbol{q}(\epsilon t))^2}, \frac{\sigma\epsilon}{\lambda_2(\boldsymbol{q}(\epsilon t))}, \frac{\sigma^2}{\lambda_2(\boldsymbol{q}(\epsilon t))}\right). \end{aligned} \tag{S.39}$$

II. RELATIONS AMONG DIFFERENT SENSITIVITY FUNCTIONS

This section gives a derivation of Eqs. (3)–(5) in the main article. These relations are essentially important in understanding the properties of the sensitivity functions and in developing methods to calculate and estimate the sensitivity functions. In this section, for simplicity of notation, the sensitivity functions are denoted by $\boldsymbol{\zeta}(\theta, \boldsymbol{I})$ and $\boldsymbol{\xi}(\theta, \boldsymbol{I})$ as in the main article.

A. Derivation of Eq. (3) in the main article

As we shown in Eq. (3) in the main article, the sensitivity function $\boldsymbol{\xi}(\theta, \boldsymbol{I})$ can be written as

$$\boldsymbol{\xi}(\theta, \boldsymbol{I}) = -\frac{\partial \boldsymbol{X}_0(\theta, \boldsymbol{I})}{\partial \boldsymbol{I}}^\top \boldsymbol{Z}(\theta, \boldsymbol{I}). \tag{S.40}$$

This equation relates the change in the shape of the limit-cycle orbit $\boldsymbol{X}_0(\theta, \boldsymbol{I})$ and the phase sensitivity function $\boldsymbol{Z}(\theta, \boldsymbol{I})$ to the sensitivity function $\boldsymbol{\xi}(\theta, \boldsymbol{I})$. From the definition of $\Theta(\boldsymbol{X}, \boldsymbol{I})$,

$$\Theta(\boldsymbol{X}_0(\theta, \boldsymbol{I}), \boldsymbol{I}) = \theta \tag{S.41}$$

holds. By differentiating Eq. (S.41) with respect to $\boldsymbol{I}$, we can obtain

$$\begin{aligned}\frac{\partial}{\partial \boldsymbol{I}} \Theta(\boldsymbol{X}_0(\theta, \boldsymbol{I}), \boldsymbol{I}) &= \frac{\partial \boldsymbol{X}_0(\theta, \boldsymbol{I})}{\partial \boldsymbol{I}}^\top \left.\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}}\right|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta, \boldsymbol{I})} + \left.\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{I}}\right|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta, \boldsymbol{I})} \\ &= \frac{\partial \boldsymbol{X}_0(\theta, \boldsymbol{I})}{\partial \boldsymbol{I}}^\top \boldsymbol{Z}(\theta, \boldsymbol{I}) + \boldsymbol{\xi}(\theta, \boldsymbol{I}) = 0, \end{aligned} \tag{S.42}$$

which leads to Eq. (S.40).

B. Derivation of Eqs. (4) and (5) in the main article

As we shown in Eqs. (4) and (5) in the main article, the sensitivity functions $\boldsymbol{\zeta}(\theta, \boldsymbol{I})$ and $\boldsymbol{\xi}(\theta, \boldsymbol{I})$ are mutually related as follows:

$$\boldsymbol{\xi}(\theta, \boldsymbol{I}) = \boldsymbol{\xi}(\theta_0, \boldsymbol{I}) - \frac{1}{\omega(\boldsymbol{I})} \int_{\theta_0}^{\theta} [\boldsymbol{\zeta}(\phi, \boldsymbol{I}) - \bar{\boldsymbol{\zeta}}(\boldsymbol{I})] d\phi, \tag{S.43}$$

$$\boldsymbol{\zeta}(\theta, \boldsymbol{I}) = \bar{\boldsymbol{\zeta}}(\boldsymbol{I}) - \omega(\boldsymbol{I}) \frac{\partial \boldsymbol{\xi}(\theta, \boldsymbol{I})}{\partial \theta}, \tag{S.44}$$

and

$$\bar{\boldsymbol{\zeta}}(\boldsymbol{I}) := \frac{1}{2\pi} \int_0^{2\pi} \boldsymbol{\zeta}(\theta, \boldsymbol{I}) d\theta = \frac{d\omega(\boldsymbol{I})}{d\boldsymbol{I}}, \tag{S.45}$$

where $\theta_0 \in [0, 2\pi)$ is an arbitrary phase and $\bar{\boldsymbol{\zeta}}(\boldsymbol{I})$ is the average of $\boldsymbol{\zeta}(\theta, \boldsymbol{I})$ with respect to θ and is a function of $\boldsymbol{I}$. Equation (S.43) (or (S.44)) represents the sensitivity function $\boldsymbol{\xi}(\theta, \boldsymbol{I})$ characterizing the phase response caused by a small constant shift in $\boldsymbol{I}$ as an integral of the phase response to the instantaneous change in $\boldsymbol{I}$ at each θ, and Eq. (S.45) relates the change in the frequency $\omega(\boldsymbol{I})$ of the limit-cycle orbit to the average of the sensitivity function $\boldsymbol{\zeta}(\theta, \boldsymbol{I})$, i.e., the net phase shift caused by the a small constant shift in $\boldsymbol{I}$ during one period of oscillation. Using these relations, we can obtain the sensitivity function $\boldsymbol{\xi}(\theta, \boldsymbol{I})$ for each $\boldsymbol{I}$. Namely, we can calculate the sensitivity function $\boldsymbol{\zeta}(\theta, \boldsymbol{I})$, e.g., by using the adjoint method, and then integrate $\boldsymbol{\zeta}(\theta, \boldsymbol{I})$ with respect to θ to obtain the sensitivity function $\boldsymbol{\xi}(\theta, \boldsymbol{I})$.

Since we can straightforwardly derive Eq. (S.43) by integrating Eq. (S.44) with respect to θ, we only describe derivations of Eq. (S.44) and Eq. (S.45). From the definition of $\Theta(\boldsymbol{X}, \boldsymbol{I})$,

$$\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}} \cdot \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{I}) = \omega(\boldsymbol{I}) \tag{S.46}$$

holds. By differentiating Eq. (S.46) with respect to $\boldsymbol{I}$ and plugging in $\boldsymbol{X} = \boldsymbol{X}_0(\theta, \boldsymbol{I})$, we can obtain

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{I}} \left[\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}} \cdot \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{I}) \right] \Bigg|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta, \boldsymbol{I})} &= \left[\frac{\partial}{\partial \boldsymbol{I}} \left(\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}} \right) \right]^\top \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{I}) \Bigg|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta, \boldsymbol{I})} \\
&\quad + \frac{\partial \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{I}}^\top \frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}} \Bigg|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta, \boldsymbol{I})} \\
&= \left[\frac{\partial}{\partial \boldsymbol{X}} \left(\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{I}} \right) \right] \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{I}) \Bigg|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta, \boldsymbol{I})} + \boldsymbol{\zeta}(\theta, \boldsymbol{I}) \\
&= \frac{d\omega(\boldsymbol{I})}{d\boldsymbol{I}},
\end{aligned} \tag{S.47}$$

where $\frac{\partial}{\partial \boldsymbol{I}}(\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}})$ is a matrix whose (i, j)-th element is given by $\frac{\partial^2 \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial X_i \partial I_j}$, and $\frac{\partial}{\partial \boldsymbol{X}}(\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{I}})$ is the transpose of $\frac{\partial}{\partial \boldsymbol{I}}(\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{X}})$. Here, the first term of the third line in Eq. (S.47) can be written as

$$\begin{aligned}
&\left[\frac{\partial}{\partial \boldsymbol{X}} \left(\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{I}} \right) \right] \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{I}) \Bigg|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta, \boldsymbol{I})} = \left[\frac{\partial}{\partial \boldsymbol{X}} \left(\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{I}} \right) \right] \Bigg|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta, \boldsymbol{I})} \frac{d\boldsymbol{X}_0(\omega(\boldsymbol{I})t, \boldsymbol{I})}{dt} \Bigg|_{t=\theta/\omega(\boldsymbol{I})} \\
&= \omega(\boldsymbol{I}) \left[\frac{\partial}{\partial \boldsymbol{X}} \left(\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{I}} \right) \right] \Bigg|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta, \boldsymbol{I})} \frac{\partial \boldsymbol{X}_0(\theta, \boldsymbol{I})}{\partial \theta} = \omega(\boldsymbol{I}) \frac{\partial}{\partial \theta} \left(\frac{\partial \Theta(\boldsymbol{X}, \boldsymbol{I})}{\partial \boldsymbol{I}} \right) \Bigg|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta, \boldsymbol{I})} \\
&= \omega(\boldsymbol{I}) \frac{\partial \boldsymbol{\xi}(\theta, \boldsymbol{I})}{\partial \theta}.
\end{aligned} \tag{S.48}$$

Then, from Eqs. (S.47) and (S.48), we can derive Eq. (S.44) and Eq. (S.45) as

$$\frac{d\omega(\boldsymbol{I})}{d\boldsymbol{I}} = \frac{1}{2\pi}\int_0^{2\pi}\left[\omega(\boldsymbol{I})\frac{\partial\boldsymbol{\xi}(\theta,\boldsymbol{I})}{\partial\theta} + \boldsymbol{\zeta}(\theta,\boldsymbol{I})\right]d\theta = \frac{1}{2\pi}\int_0^{2\pi}\boldsymbol{\zeta}(\theta,\boldsymbol{I})d\theta, \tag{S.49}$$

where the first term in the integral vanishes due to 2π-periodicity of $\boldsymbol{\xi}(\theta,\boldsymbol{I})$.

III. RELATION BETWEEN THE CONVENTIONAL AND GENERALIZED PHASE EQUATIONS

Here we compare the generalized phase equation with the conventional phase equation using the near-identity transformation. As stated in the main article, the conventional phase equation can be written as

$$\dot{\tilde{\theta}} = \omega(\boldsymbol{q}_c) + \sigma_c\boldsymbol{\zeta}(\tilde{\theta},\boldsymbol{q}_c)\cdot\boldsymbol{p}_c(t) + O(\sigma_c^2), \tag{S.50}$$

where $\boldsymbol{q}_c \in A$ is a constant, $\boldsymbol{p}_c(t)$ is an external input defined as $\sigma_c\boldsymbol{p}_c(t) = \boldsymbol{I}(t) - \boldsymbol{q}_c$, and σ_c is a parameter representing the intensity of the external input. We decompose the external input $\boldsymbol{p}_c(t)$ into two terms, $\boldsymbol{p}_1(t)$ and $\boldsymbol{p}_2(t)$, as

$$\boldsymbol{p}_c(t) = \boldsymbol{p}_1(t) + \boldsymbol{p}_2(t), \tag{S.51}$$

and introduce a slightly deformed phase $\phi(t)$ as

$$\phi(t) = \tilde{\theta}(t) + \sigma_c\boldsymbol{\xi}(\tilde{\theta}(t),\boldsymbol{q}_c)\cdot\boldsymbol{p}_1(t). \tag{S.52}$$

By applying the above near-identity transformation to the phase equation (S.50), we can derive the following phase equation for $\phi(t)$:

$$\dot{\phi} = \omega(\boldsymbol{q}_c) + \sigma_c\left.\frac{d\omega(\boldsymbol{I})}{d\boldsymbol{I}}\right|_{\boldsymbol{I}=\boldsymbol{q}_c}\cdot\boldsymbol{p}_1(t) + \sigma_c\boldsymbol{\zeta}(\phi,\boldsymbol{q}_c)\cdot\boldsymbol{p}_2(t) + \sigma_c\boldsymbol{\xi}(\phi,\boldsymbol{q}_c)\cdot\dot{\boldsymbol{p}}_1(t) + O(\sigma_c^2), \tag{S.53}$$

where we used Eqs. (S.44) and (S.45). Without loss of generality, we can regard the input terms $\sigma_c\boldsymbol{p}_1(t)$ and $\sigma_c\boldsymbol{p}_2(t)$ in Eq. (S.53) as the slowly varying part $\boldsymbol{q}(\epsilon t)$ and the weak fluctuations $\sigma\boldsymbol{p}(t)$ in the main article, because we can choose the decomposition of $\boldsymbol{p}_c(t)$ arbitrarily. Then, Eq. (S.53) can be considered an approximation to the generalized phase equation (2) in the main article. In other words, the first term of Eq. (S.53) represents the first-order (linear) approximation in $\boldsymbol{q}$ around $\boldsymbol{q} = \boldsymbol{q}_c$ to the first term of the generalized phase equation, while the second and third terms of Eq. (S.53) are zeroth-order (constant) approximations in $\boldsymbol{q}$ around $\boldsymbol{q} = \boldsymbol{q}_c$ to the second and third terms of the generalized phase equation.

In this sense, the generalized phase equation (2) in the main article can be considered a nonlinear generalization of the conventional phase equation (S.53). For the modified Stuart-Landau oscillator defined in the main article, the frequency $\omega(I)$ and the sensitivity functions $\zeta(\theta, I)$ and $\xi(\theta, I)$ are explicitly given by

$$\omega(I) = e^{2I} = 1 + 2I + 2I^2 + \frac{4I^3}{3} + \cdots, \tag{S.54}$$

$$\zeta(\theta, I) = 2e^{2I} - e^{I}\cos\theta = 1 - \cos\theta + (4 - \cos\theta)I + \left(4 - \frac{\cos\theta}{2}\right)I^2 + \cdots, \tag{S.55}$$

$$\xi(\theta, I) = e^{-I}\sin\theta = \sin\theta - (\sin\theta)I + \frac{\sin\theta}{2}I^2 + \cdots. \tag{S.56}$$

When the temporal variation in the input $I(t)$ is sufficiently small, we can truncate $\omega(I)$ at the first order, and $\zeta(\theta, I)$ and $\xi(\theta, I)$ at the zeroth order, which is equivalent to using the conventional phase equation. However, when the input $I(t)$ varies largely with time and the shape of the limit-cycle orbit is significantly deformed, the above approximation is no longer valid. In such cases, the conventional phase equation would fail to predict the actual oscillator dynamics and the generalized phase reduction method should be used.

IV. ACCURACY AND ROBUSTNESS OF THE GENERALIZED PHASE EQUATION

In the main article, we briefly demonstrated that the generalized phase equation can accurately predict the time series of the oscillation phase as compared to the conventional phase equation. Here, we examine the accuracy and robustness of the generalized phase equation in more detail with numerical simulations. We use a modified Stuart-Landau oscillator defined as

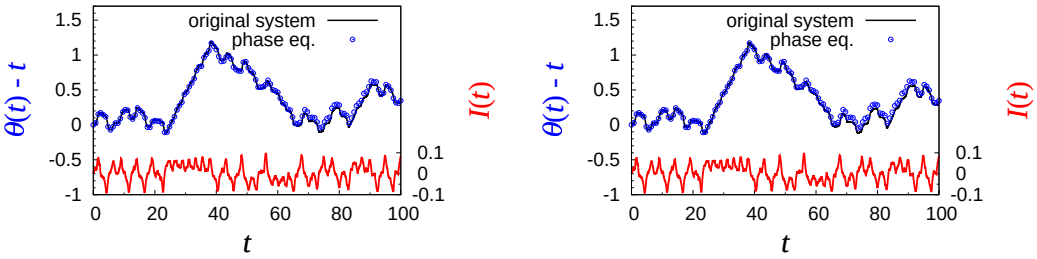
$$\dot{x} = e^{2I(t)}(\lambda_0 x - y - \lambda_0 I(t)) - \lambda_0[(x - I(t))^2 + y^2](x - I(t)), \tag{S.57}$$

$$\dot{x} = e^{2I(t)}(x + \lambda_0 y - I(t)) - \lambda_0[(x - I(t))^2 + y^2]y, \tag{S.58}$$

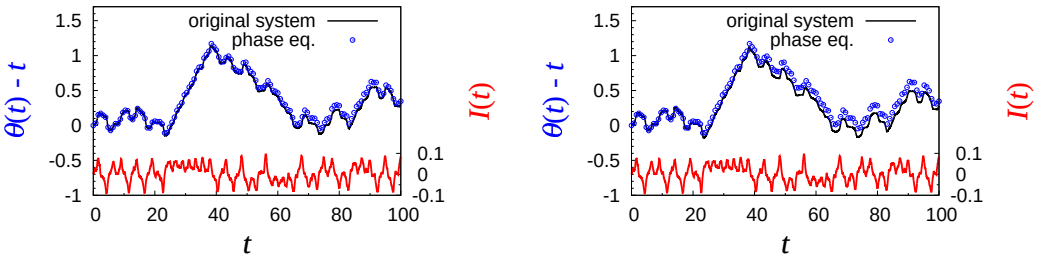
whose amplitude relaxation rate can explicitly be specified by the parameter λ_0. Here, x and y are state variables representing the oscillator state, $I(t)$ is an external input, and λ_0 is a parameter that controls the timescale of the amplitude relaxation. For this model, one can explicitly define the amplitude $r = \sqrt{(x - I(t))^2 + y^2}$, which decays exponentially as $\dot{r} = -2\lambda_0 r$. As stated in the main article, the small parameter ϵ represents the relative timescale of the slowly varying component $\boldsymbol{q}(\epsilon t)$ to the amplitude relaxation time of the oscillator (which was assumed to be $O(1)$ in the main article). Thus, by varying the parameter λ_0, we can effectively control the parameter ϵ.

We applied a periodically varying parameter

$$I(t) = 0.005L_1(0.3t) + \sigma L_2(t) \tag{S.59}$$



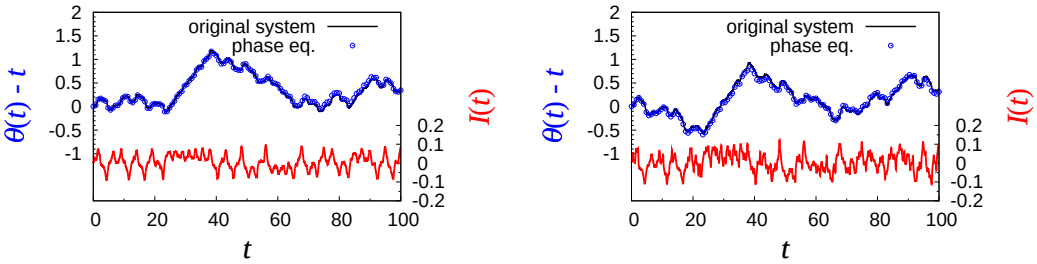
(a) $\lambda_0 = 1.000$, $\sigma = 0.001$



(b) $\lambda_0 = 0.215$, $\sigma = 0.001$

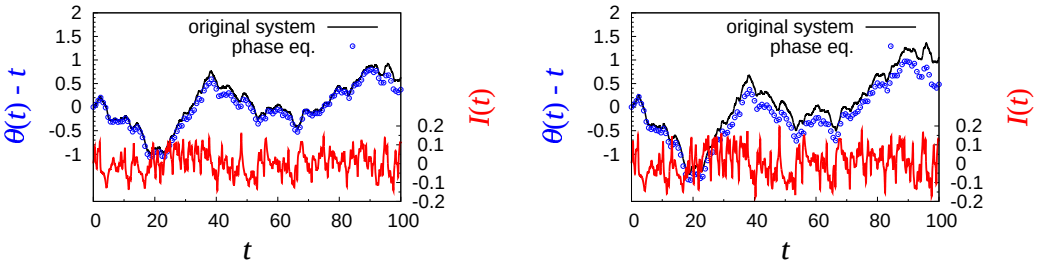
(c) $\lambda_0 = 0.046$, $\sigma = 0.001$

(d) $\lambda_0 = 0.010$, $\sigma = 0.001$



(e) $\lambda_0 = 1.000$, $\sigma = 0.001$

(f) $\lambda_0 = 1.000$, $\sigma = 0.003$



(g) $\lambda_0 = 1.000$, $\sigma = 0.005$

(h) $\lambda_0 = 1.000$, $\sigma = 0.007$

FIG. 1. (Color online) Accuracy and robustness of the generalized phase equation. A modified Stuart-Landau oscillator (Eqs. (S.57) and (S.58)) is driven by a periodically varying parameter $I(t)$ (red lines, Eq. (S.59)). The time series of the phase $\theta(t) = \Theta(\boldsymbol{X}(t), \boldsymbol{q}(\epsilon t))$ measured directly from the original system (black lines) and predicted by the direct numerical simulation of the generalized phase equation (blue circles) are plotted. In (a)–(d), σ is fixed at 0.001 and λ_0 is varied between 1 and 0.01, while in (e)–(h), λ_0 is fixed at 1 and σ is varied between 0.001 and 0.007.

to the oscillator, where $L_1(t)$ and $L_2(t)$ are independently generated time series of the variable x of the chaotic Lorenz model [2], $\dot{x} = 10(y - x)$, $\dot{y} = x(28 - z) - y$, and $\dot{z} = xy - 8z/3$, and σ is a parameter controlling the intensity of the high-frequency components in $I(t)$. Since the parameters λ_0 and σ play important roles in the proposed phase reduction method, we examine the accuracy and robustness of the generalized phase equation for varying values of λ_0 and σ.

Figure 1 shows the results of numerical simulations, where one of the parameters is kept fixed and the other is varied. In Figs. 1 (a)–(d), σ is fixed and λ_0 is varied. The accuracy of the proposed phase reduction method is deteriorated as λ_0 is decreased. In this case, when $\lambda_0 > 0.01$, the generalized phase equation can predict the temporal evolution of the actual phase of the oscillator. Similarly, when λ_0 is fixed and σ is varied (Figs. 1 (e)–(h)), the accuracy of the proposed method becomes worse as σ is increased. In this case, when $\sigma < 0.007$, the generalized phase equation can predict the temporal evolution of the actual phase.

V. PHASE LOCKING OF THE MORRIS-LECAR MODEL DRIVEN BY STRONG PERIODIC FORCING

In the main article, we analyzed the phase locking of a modified Stuart-Landau oscillator to periodic forcing and demonstrated the usefulness of the proposed phase reduction method. In this section, we further analyze another type of limit-cycle oscillator, i.e., the Morris-Lecar model [3], which describes periodic firing of a neuron. We theoretically analyze the phase locking dynamics of the Morris-Lecar model to periodic external forcing and compare the theoretical predictions with direct numerical simulations.

A. The Morris-Lecar model

The Morris-Lecar model [3] of a periodically firing neuron has a two-dimensional state variable $\boldsymbol{X}(t) = [V(t), w(t)]^\top$. The vector field $\boldsymbol{F}(\boldsymbol{X}, I) = [F_1(V, w, I), F_2(V, w, I)]^\top$ is given by

$$C_{\mathrm{m}} F_1 = g_{\mathrm{L}}(V_L - V) + g_{\mathrm{K}} w (V_K - V) + g_{\mathrm{Ca}} m_\infty (V_{\mathrm{Ca}} - V) + I, \tag{S.60}$$

$$F_2 = \lambda_w (w_\infty - w), \tag{S.61}$$

where $m_\infty(V) = 0.5\{1 + \tanh[(V - V_1)/V_2]\}$ and $w_\infty(V) = 0.5\{1 + \tanh[(V - V_3)/V_4]\}$ are the conductance functions, I is the parameter to which the forcing is applied, and V_{K}, V_{L}, V_{Ca}, g_{K}, g_{L}, g_{Ca}, C, V_1, V_2, V_3, V_4, and λ_w are constant parameters. This model exhibits stable limit-cycle oscillations when the parameter values are chosen appropriately.

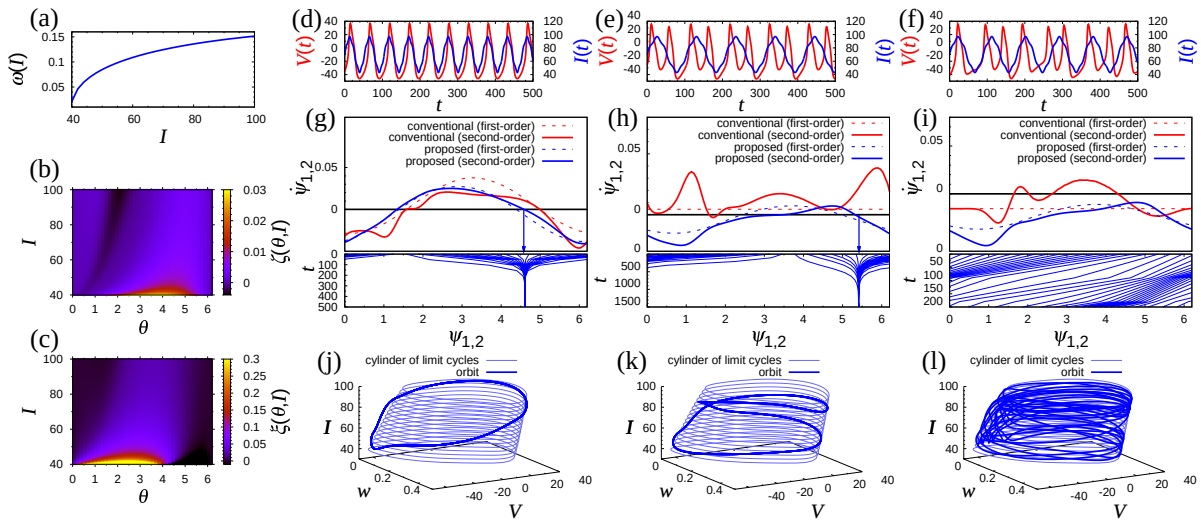


FIG. 2. (Color online) Phase locking of the Morris-Lecar model exhibiting smooth oscillations. Three sets of periodically varying parameters, $I^{(j)}(t)$: $q^{(j)}(\epsilon t) = 70 + 25\sin(\omega_I^{(j)} t)$ and $\sigma p^{(j)}(t) = 2\sin(5\omega_I^{(j)} t)$ with $\omega_I^{(1,2,3)} = 0.12, 0.06, 0.07$ are used, which lead to $1:1$ or $1:2$ phase locking to $q(\epsilon t)$; $1:1$ phase locking to $I^{(1)}(t)$ [(d), (g), and (j)], $1:2$ phase locking to $I^{(2)}(t)$ [(e), (h), and (k)], and failure of phase locking to $I^{(3)}(t)$ [(f), (i), and (l)]. (a) Natural frequency $\omega(I)$. (b), (c) Sensitivity functions $\zeta(\theta, I)$ and $\xi(\theta, I)$. (d)–(f) Time series of the state variable $V(t)$ of a periodically driven oscillator (red) and the periodic external forcing (blue). (g)–(i) Dynamics of the phase difference ψ. The averaged dynamics of ψ is shown in the top panel, where the stable phase difference predicted by the second-order averaging of the generalized phase equation is indicated by an arrow, and evolution of ψ from 20 different initial states are plotted in the bottom panel. (j)–(l) Orbits of the periodically driven oscillator (blue) and I-dependent stable limit-cycle solutions (light blue) plotted in three-dimensional space (V, w, I).

B. Smooth oscillations

We set the parameters as $V_{\rm K} = -84$, $V_{\rm L} = -60$, $V_{\rm Ca} = 120$, $g_{\rm K} = 8$, $g_{\rm L} = 2$, $g_{\rm Ca} = 4$, $C = 20$, $V_1 = -1.2$, $V_2 = 18$, $V_3 = 12$, $V_4 = 17$, and $\lambda_w = 0.0667$. For these parameters, a stable limit cycle emerges via a saddle-node on invariant circle (SNIC) bifurcation at $I \simeq 50$, and vanishes via a Hopf bifurcation at $I \simeq 115$. The oscillation remains generally smooth for all values of I. The phase sensitivity function has the type-I shape with a positive lobe near the SNIC bifurcation, and a sinusoidal type-II shape with both positive and negative lobes near the Hopf bifurcation [3]. Thus, when the external forcing $I(t)$ is time-varying, the shape of the orbit, frequency, and phase response properties of the oscillator can vary significantly with time.

Numerically calculated $\omega(I)$, $\zeta(\theta, I)$, and $\xi(\theta, I)$ are shown in Figs. 2 (a)–(c), and phase-locked dynamics of the variable $V(t)$ to the periodic forcing $I(t)$ is shown in Figs. 2(d)–(f). Note that the

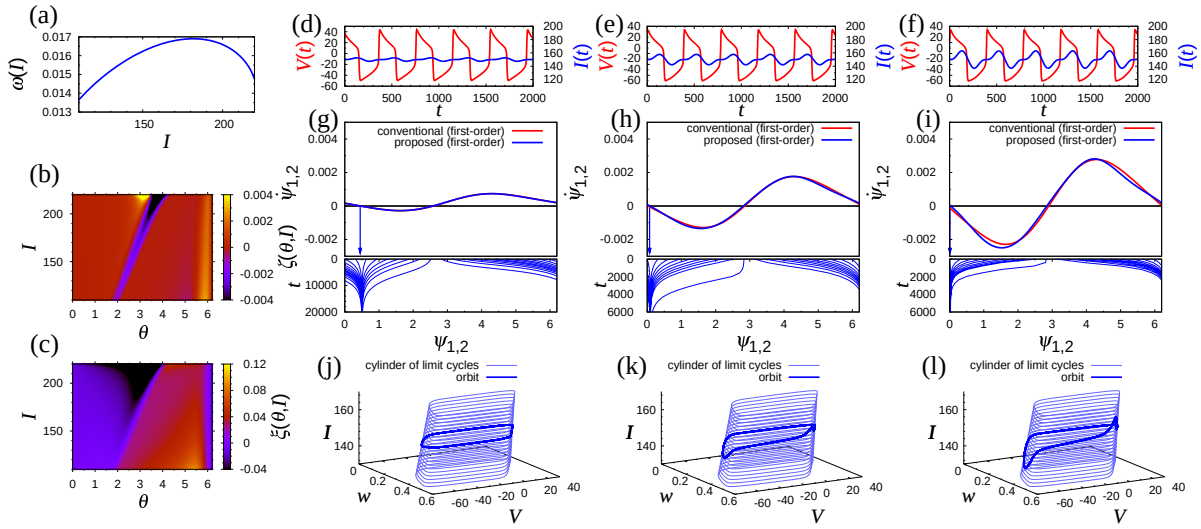


FIG. 3. (Color online) Phase locking of the Morris-Lecar model (relaxation oscillation). Three types of periodically varying parameters, $I^{(j)}(t)$: $q^{(j)}(\epsilon t) = 150 + \alpha^{(j)}\sin(\omega_I t) - \alpha^{(j)}\sin(2\omega_I t)$ and $\sigma p^{(j)}(t) = 0$ with $\alpha^{(4,5,6)} = 10$, 15, 20 and $\omega_I = 0.016$ are used, which lead to 1 : 1 phase locking to $I^{(4)}(t)$ [(d), (g), and (j)], $I^{(5)}(t)$ [(e), (h), and (k)], and $I^{(6)}(t)$ [(f), (i), and (l)]. (a) Natural frequency $\omega(I)$. (b), (c) Sensitivity functions $\zeta(\theta, I)$ and $\xi(\theta, I)$. (d)–(f) Time series of the state variable $V(t)$ of a periodically driven oscillator (red) and periodic external forcing (blue). (g)–(i) Dynamics of the phase difference ψ with an arrow representing the stable phase difference (top panel) and evolution of ψ from 20 different initial states (bottom panel). (j)–(l) Orbits of a periodically driven oscillator (blue) and I-dependent stable limit-cycle solutions (light blue) plotted in three-dimensional space (V, w, I).

oscillations are significantly deformed due to strong periodic forcing. Figures 2 (g)–(i) compare the results of the reduced phase equations with those of the direct numerical simulations. We can confirm that the generalized phase reduction theory nicely predicts the stable phase differences ψ, while the conventional method does not. The orbits of the oscillator and the cylinder C of the limit cycles in three-dimensional space (V, w, I) are plotted in Figs. 2 (j)–(l), showing synchronous [(j) and (k)] or asynchronous (l) dynamics with the periodic forcing.

C. Relaxation oscillations

We set the parameters as $V_{\rm K} = -84$, $V_{\rm L} = -60$, $V_{\rm Ca} = 120$, $g_{\rm K} = 8$, $g_{\rm L} = 2$, $g_{\rm Ca} = 4.4$, $C = 20$, $V_1 = -1.2$, $V_2 = 18$, $V_3 = 2$, $V_4 = 30$, and $\lambda_w = 0.004$. For these parameters, the ML model exhibits relaxation oscillations consisting of fast and slow dynamics in an appropriate range of I, and correspondingly the phase sensitivity function takes an impulse-like shape. Numerically

calculated $\omega(I)$, $\zeta(\theta, I)$, and $\xi(\theta, I)$ are shown in Figs. 3 (a)–(c), and the phase-locked dynamics of $V(t)$ to the periodic forcing $I(t)$ are shown in Figs. 3 (d)–(f). Figures 3 (g)–(i) compare the results of the reduced phase equations with those of the direct numerical simulations. The parameter I was varied between 140 and 200. In this case, both the conventional and generalized phase equations seem to nicely predict the stable phase difference. As shown below, however, the conventional phase equation may actually fail to predict the oscillator dynamics in such cases.

To investigate whether the two phase equations can accurately predict dynamics of the original limit-cycle oscillator, we calculate the phase maps [2], corresponding to the numerical simulations shown in Fig. 3. The phase map is a one-dimensional map from the phase $\theta(nT_I)$ at $t = nT_I$ to the phase $\theta((n+1)T_I)$ after one period of the external forcing, where $n \in \mathbb{N}$ is an integer and T_I is the period of external forcing. Figure 4 compares the phase maps calculated by direct numerical simulations of the original limit-cycle oscillator with those obtained by the conventional and generalized phase equations. These results indicate that the generalized phase equation well captures the dynamics of the oscillator, while the conventional equation does not; it turns out that the conventional phase equation could not actually predict the oscillator dynamics in the numerical simulation of Fig. 3, and the seemingly correct prediction of the stable phase difference was a coincidence.

D. Significantly deformed oscillations

As a further confirmation of the validity of the generalized phase equation, we performed numerical simulations using the Morris-Lecar model whose limit-cycle orbit is significantly deformed by the variations in the parameter $I(t)$. In this case, the oscillator is driven by a multiplicative input term, $V \cdot I(t)$, as

$$C_{\mathrm{m}} F_1 = g_{\mathrm{L}}(V_L - V) + g_{\mathrm{K}} w (V_K - V) + g_{\mathrm{Ca}} m_\infty (V_{\mathrm{Ca}} - V) + V \cdot I(t) + 150, \tag{S.62}$$

$$F_2 = \lambda_w (w_\infty - w), \tag{S.63}$$

where all parameters are the same as in Fig. 3 (relaxation oscillations). Here, the input $I(t)$ to the neuron corresponds to the conductance of the membrane, rather than to the injected current [3]. Numerically calculated $\omega(I)$, $\zeta(\theta, I)$, and $\xi(\theta, I)$ are shown in Figs. 5 (a)–(c), and the phase-locked dynamics of $V(t)$ to the periodic forcing $I(t)$ are shown in Figs. 5 (d)–(f). Figure 6 compares the phase maps calculated by direct numerical simulations of the original limit-cycle oscillator with those obtained by the conventional and generalized phase equations. These result show that the

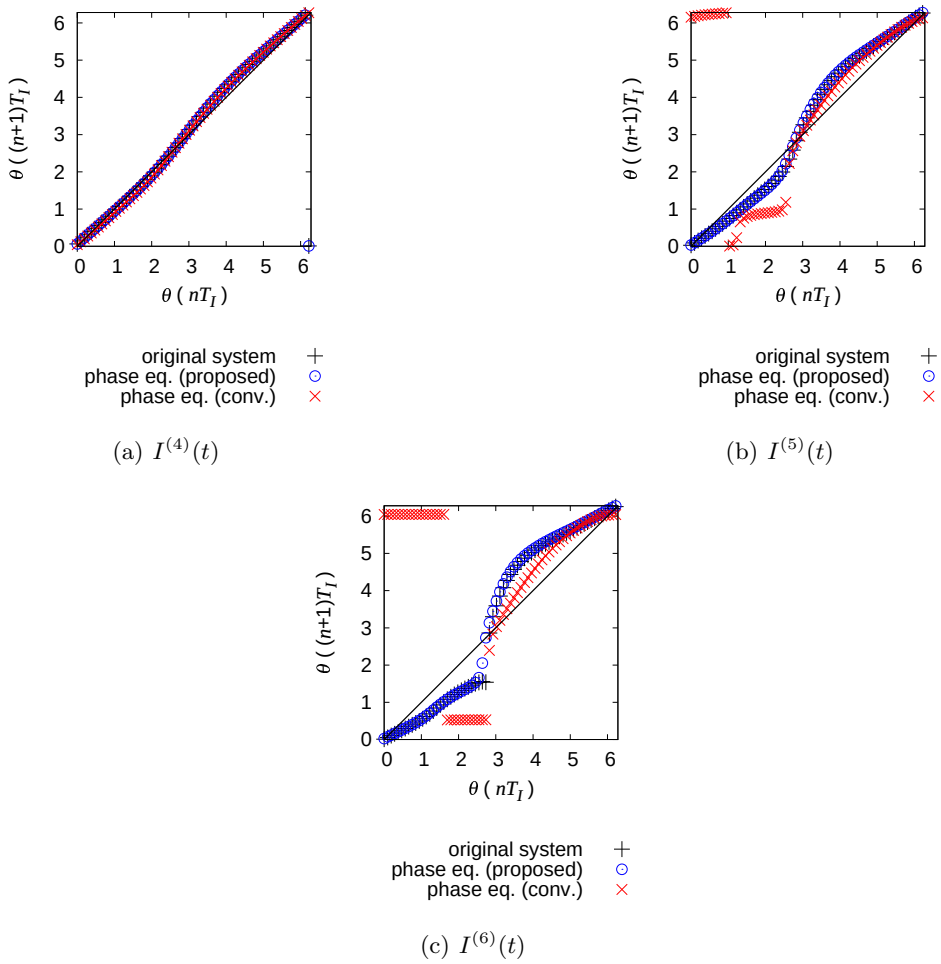


FIG. 4. (Color online) Phase maps calculated by direct numerical simulations of the original limit-cycle oscillator (black crosses) and by the conventional (red crosses) and generalized (blue circles) phase equations. Results for the three types of the periodic forcing used in Fig. 3, i.e., (a) $I^{(4)}(t)$, (b) $I^{(5)}(t)$, and (c) $I^{(6)}(t)$, are shown.

generalized phase equation remains valid even when the limit-cycle orbit is significantly deformed by the largely varying parameter, while the conventional method does not.

---

[1] D. S. Goldobin, J. Teramae, H. Nakao, and G. B. Ermentrout, Phys. Rev. Lett. **105**, 154101 (2010).

[2] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, 2001).

[3] G. B. Ermentrout and D. H. Terman, *Mathematical Foundations of Neuroscience* (Springer, New York, 2010).

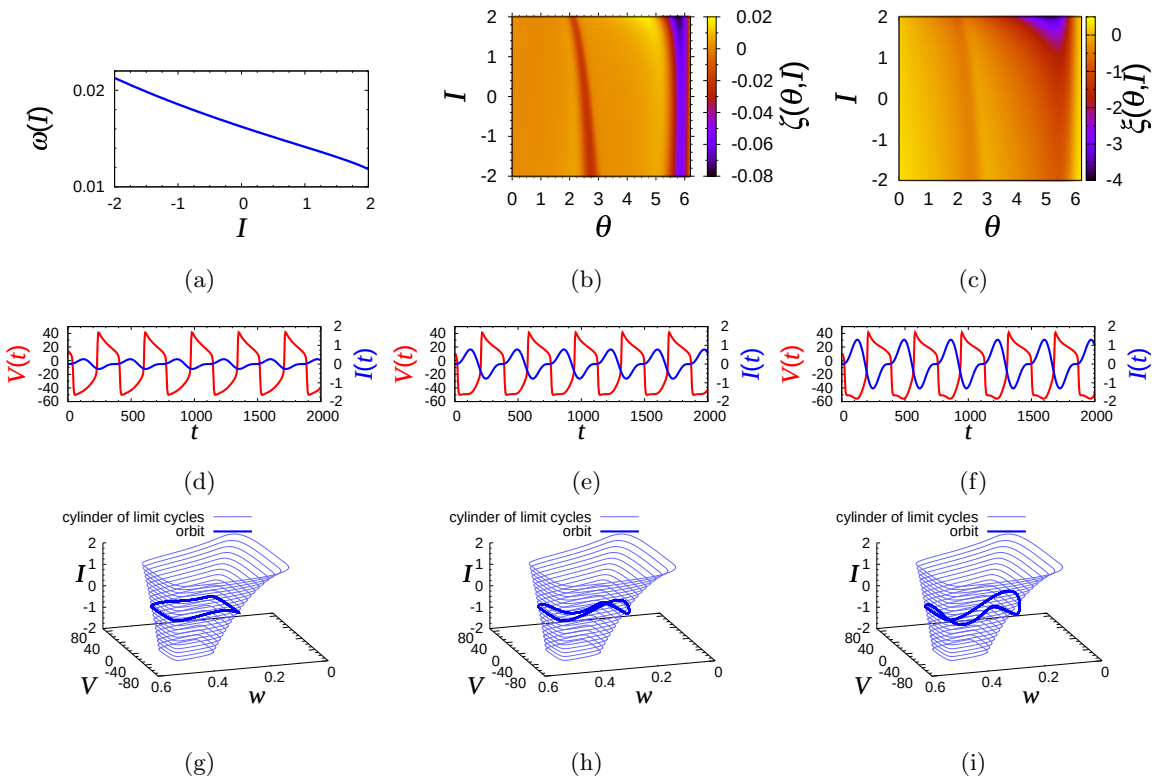
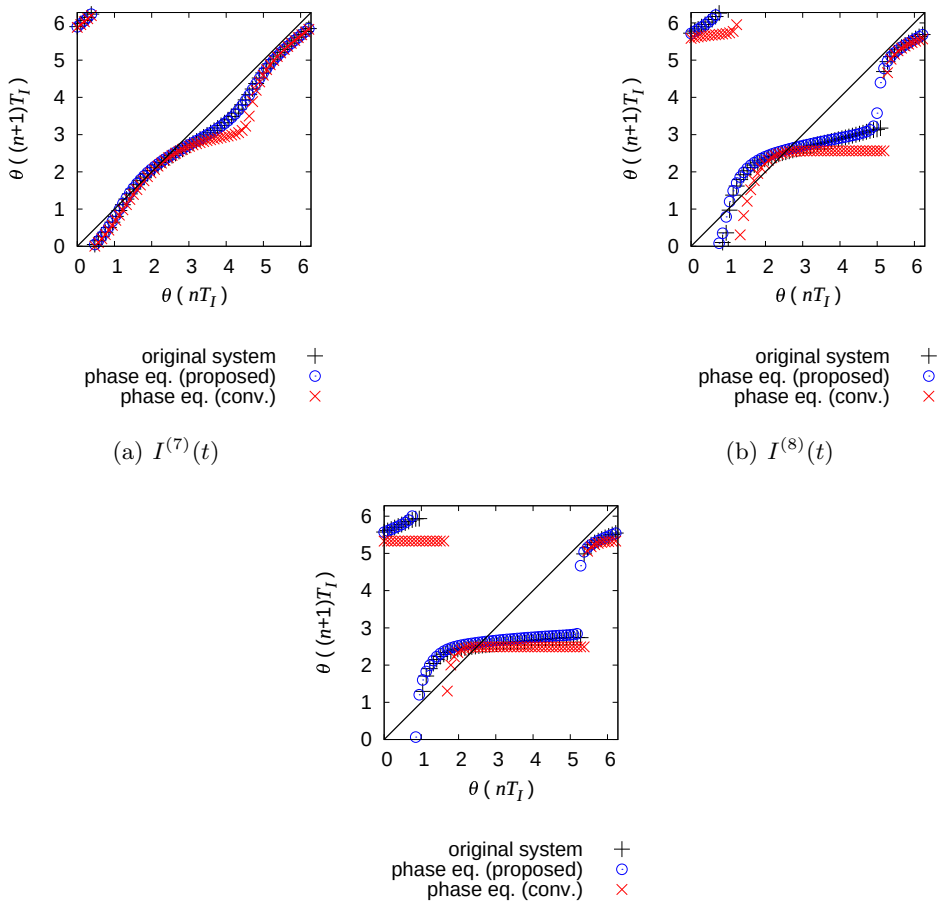


FIG. 5. (Color online) Phase locking of the Morris-Lecar model (with significantly deformed limit-cycle orbit). Three types of periodically varying parameters, $I^{(j)}(t)$: $q^{(j)}(\epsilon t) = \alpha^{(j)}\sin(\omega_I t) - \frac{\alpha^{(j)}}{2}\sin(2\omega_I t)$ and $\sigma p^{(j)}(t) = 0$ with $\alpha^{(7,8,9)} = 0.2$, 0.6, 1.0 and $\omega_I = 0.017$ are used, which lead to $1:1$ phase locking to $I^{(7)}(t)$ [(d) and (g)], $I^{(8)}(t)$ [(e) and (h)], and $I^{(9)}(t)$ [(f) and (i)]. (a) Natural frequency $\omega(I)$. (b), (c) Sensitivity functions $\zeta(\theta, I)$ and $\xi(\theta, I)$. (d)–(f) Time series of the state variable $V(t)$ of a periodically driven oscillator (red) and periodic external forcing (blue). (g)–(i) Orbits of a periodically driven oscillator (blue) and I-dependent stable limit-cycle solutions (light blue) plotted in three-dimensional space (V, w, I).



(a) $I^{(7)}(t)$

(b) $I^{(8)}(t)$

(c) $I^{(9)}(t)$

FIG. 6. (Color online) Phase maps calculated by direct numerical simulations of the original limit-cycle oscillator (black crosses) and by the conventional (red crosses) and generalized (blue circles) phase equations. Results for the three types of the periodic forcing used in Fig. 5, i.e., (a) $I^{(7)}(t)$, (b) $I^{(8)}(t)$, and (c) $I^{(9)}(t)$, are shown.